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AN EXTENSION OF GAUSS'S PRINCIPLE OF LEAST CONSTRAINT

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Gauss's principle of least constraint is reformulated to cover cases in which the constraint forces may do positive or negative work on a system in a virtual displacement. This is needed to deal conveniently with cases in which, for example, friction is significant.

Keywords: Mechanical systems, Gauss's principle of least constraint, constraint forces, Lagrange multiplier

1. INTRODUCTION

Lagrange and Gauss were among the first system scientists. They considered mechanical systems as being characterized by potential energy, kinetic energy and the constraint function, all of which refer to the system as a whole. Lagrange also emphasized the use of generalized coordinates to describe the current configuration. He focused attention on the need to consider constraints on the system, and introduced what are today called Lagrange multipliers. Constraints are what make a collection of point masses and rigid bodies into a system.

Lagrange proposed a fundamental principle for dealing with the constraints, the principle of virtual work: the constraint forces do no work on the system in a virtual displacement. This has worked well because practical mechanical systems, through design and the use of lubricants, minimize the effects of constraint forces that do work on a system. Using this principle, Gauss was able, in 1829, to establish his principle of least constraint, which provided an alternative variational principle for the motion of mechanical systems.

The aim of this paper is to produce an extension of Gauss's principle which is applicable in situations where constraint forces such as friction do work on the system in virtual displacements.

A key ingredient in the analysis is the employment of pseudoinverses of matrices, a tool that was not available to those early investigators. In Section 2, we sketch Lagrange's and Gauss's approach to general constrained motion of mechanical systems and state the result

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that they lead to for the actual acceleration vector (Eq. (8)). Section 3 goes on to state an extension of the principle of virtual work to cover cases in which there are constraint forces that do work on the system. Then in Section 4 the extended principle of virtual work is used to derive the new equation of motion (Eq. (33)). This makes it possible to provide an extension of Gauss's principle in Section 5, one which covers the case in which constraint forces that do work on the system in a virtual displacement are significant. The validity of the new principle is verified by showing that it leads to the same equation of motion, Eq. (44), as was established earlier in Eq. (33).

Equation (44) is the most general possible equation of constrained motion. In modeling a mechanical system the modeler may neglect constraint forces that do work on the system by putting $c = 0$. Or they may be included by having $c \neq 0$ and specified by Eq. (12). There are no remaining possibilities, as the derivation reveals.

2. CLASSICAL MODELS OF CONSTRAINED MOTION

Consider a discrete mechanical system whose generalized state vector is q, generalized velocity vector is $\dot{q}$, and generalized acceleration is $\ddot{q}$. These vectors are of dimension n. Then Lagrange's equations of motion take the form

$$M\ddot{q} = Q, \tag{1}$$

where M is an n by n positive definite symmetric matrix, and Q is an n-dimensional force vector. Suppose next that this system is subjected to m consistent equality constraints of the form

$$f_i(t, q, \dot{q}) = 0, \quad i = 1, 2, \ldots, m. \tag{2}$$

Upon differentiation with respect to t these equations assume the form

$$A\ddot{q} = b, \tag{3}$$

where A is an m by n rectangular matrix, and b is an m-dimensional vector. Both may depend upon t, q, $\dot{q}$. To maintain these constraints a constraint force Q^{c} is required so that Eq. (1) becomes

$$M\ddot{q} = Q + Q^{\mathrm{c}}, \tag{4}$$

where the constraint force Q^{c} is n-dimensional.

For the determination of the unknown vectors $\ddot{q}$ and Q^{c} at each moment, $2n$ independent relations are required. Equation (4) provides n, and if the rank of A is r, Eq. (3) provides r more. A customary way of providing the additional ones is through the principle of virtual work. If we let v be an n-dimensional vector such that

$$Av = 0, \tag{5}$$

where v is a so-called generalized virtual displacement, then the principle is that the constraint force Q^{c} does no work in such a virtual displacement. This provides the additional $n-r$ conditions needed for the determination of both $\ddot{q}$ and Q^{c}, which we shall actually see below.

In equation form this is: if v is a solution of $Av = 0$, then

$$v^{\mathrm{T}}Q^{\mathrm{c}} = 0. \tag{6}$$

Alternatively, Gauss, in 1829, suggested determining $\ddot{q}$ as the vector which minimizes the constraint function G,

$$G = (\ddot{q} - a)^{\mathrm{T}} M(\ddot{q} - a), \tag{7}$$

subject to the constraint $A\ddot{q} = b$, where a is the acceleration that the system would have if there were no constraints. Both of these characterizations lead to the explicit formula for $\ddot{q}$ as

$$\ddot{q} = a + M^{-1/2}\left(AM^{-1/2}\right)^{+}(b - Aa), \tag{8}$$

where $(AM^{-1/2})^{+}$ is the usual pseudoinverse of the matrix $AM^{-1/2}$. This formula (Udwadia and Kalaba, 1996) shows that the actual acceleration is made up of the free motion acceleration plus a perturbation term.

This term is proportional to the vector $b - Aa$, which indicates the extent to which the free motion acceleration a does not satisfy the constraint equation $A\ddot{q} = b$. The proportionality matrix is $M^{-1/2}(AM^{-1/2})^{+}$. The formula in Eq. (8) is useful theoretically and especially computationally, for computing environments such as MATLAB contain commands for obtaining pseudoinverses.

3. INCORPORATION OF CONSTRAINT FORCES THAT DO WORK

But suppose that still other constraint forces are at work, forces that may do positive or negative work on the system in a virtual displacement. Their existence is beyond dispute. How may the standard framework be modified to account for them?

Let us simply replace Eq. (6) with the relation

$$v^{\mathrm{T}} Q^{\mathrm{c}} = v^{\mathrm{T}} c, \tag{9}$$

where c is an n-dimensional vector which may depend upon t, q, and $\dot{q}$. The vector c, a force, could, for example, represent the force of friction. The right side of the last equation, which may be positive, zero, or negative, can be chosen to account for the forces that *do* do work on the system in a virtual displacement. Now let us see what effect this has on both Gauss's principle and the explicit formula (8) for the actual generalized acceleration $\ddot{q}$.

4. DERIVATION OF THE NEW EQUATION OF MOTION

The model for determining $\ddot{q}$ and Q^{c} is now

$$M\ddot{q} = Q + Q^{\mathrm{c}}, \tag{10}$$

$$A\ddot{q} = b, \tag{11}$$

and, for all v such that $Av = 0$,

$$v^{T} Q^{\mathrm{c}} = v^{T} c. \tag{12}$$

We introduce the notation

$$B = AM^{-1/2}, \tag{13}$$

$$\ddot{r} = M^{1/2}\ddot{q}, \tag{14}$$

so that Eq. (11) becomes

$$B\ddot{r} = b, \tag{15}$$

the most general solution of which is

$$\ddot{r} = B^{+}b + (I - B^{+}B)w. \tag{16}$$

In the last equation w is an arbitrary n-vector. Equations (10) and (12) yield

$$v^{\mathrm{T}}(M\ddot{q} - Q) = v^{\mathrm{T}}c, \tag{17}$$

for all v for which $Av = 0$. This may be rewritten as

$$v^{\mathrm{T}}(M\ddot{q} - Q - c) = 0, \tag{18}$$

or

$$v^{\mathrm{T}}(M^{1/2}\ddot{r} - Q - c) = 0. \tag{19}$$

Making use of Eq. (16) it is seen that

$$v^{\mathrm{T}}\left(M^{1/2}\{B^{+}b + (I - B^{+}B)w\} - Q - c\right) = 0. \tag{20}$$

We may rewrite the equation $Av = 0$ as

$$\left(AM^{-1/2}\right)\left(M^{1/2}v\right) = 0, \tag{21}$$

and put

$$u = M^{1/2}v, \tag{22}$$

so that

$$u^{\mathrm{T}} = v^{\mathrm{T}}M^{1/2} \tag{23}$$

and

$$Bu = 0. \tag{24}$$

Equation (20) becomes

$$u^{\mathrm{T}}\left\{B^{+}b + (I - B^{+}B)w - M^{-1/2}Q - M^{-1/2}c\right\} = 0. \tag{25}$$

Since the vector u is in the null space of the matrix B, $Bu = 0$, it is known that

$$u^{\mathrm{T}}B^{+} = 0, \tag{26}$$

which means that

$$u^{\mathrm{T}}\left\{w - M^{-1/2}Q - M^{-1/2}c\right\} = 0. \tag{27}$$

Since the term in brackets is orthogonal to every vector u for which $Bu = 0$, it must belong to the range space of the matrix B^{T}, so that

$$w - M^{-1/2}Q - M^{-1/2}c = B^{\mathrm{T}}z, \tag{28}$$

where z is an m-dimensional vector. Equation (16) becomes

$$\ddot{r} = B^{+}b + \left(I - B^{+}B\right)\left[M^{-1/2}Q + M^{-1/2}c + B^{\mathrm{T}}z\right]. \tag{29}$$

But it is known that

$$B^{+}BB^{\mathrm{T}} = B^{\mathrm{T}}, \tag{30}$$

so that the last equation becomes

$$\ddot{r} = B^{+}b + \left(I - B^{+}B\right)\left[M^{-1/2}Q + M^{-1/2}c\right]. \tag{31}$$

It follows that

$$M^{1/2}\ddot{q} = B^{+}b + \left(I - B^{+}B\right)\left[M^{-1/2}Q + M^{-1/2}c\right], \tag{32}$$

which implies that

$$\ddot{q} = a + M^{-1/2}B^{+}(b - Aa) + M^{-1/2}(I - B^{+}B)M^{-1/2}c, \tag{33}$$

where $a = M^{-1}Q$. This is the desired equation of motion. It is the most general one possible, as the derivation shows. The first two terms on the right side have already been seen in Eq. (8). The new term, due to the constraint forces that do work on the system in a virtual displacement, is the third one. Thus only a small change is needed to obtain the most general equation of constrained motion.

Equation (31) is especially informative. The first term on the right side, $B^{+}b$, is a vector in the range space of the matrix B^{T} and represents the contribution of the constraint forces that do no work on the system. The second term on the right is a vector that lies in the null space of the matrix B. It depends linearly on the generalized impressed force vector Q and the vector c, which characterizes the constraint forces that do work on the system. The first and second terms on the right side of Eq. (31) are, of course, orthogonal to each other, and one cannot compensate for the other.

Equations (31) and (33) are useful in dealing with Coulomb sliding friction. If there is a single particle, the vector c, assuming Cartesian coordinates, lies in the direction of $-\dot{q}/\left(q^{\mathrm{T}}q\right)^{1/2}$ and has a magnitude that depends upon the vector $B^{+}(b - BM^{-1/2}Q) = B^{+}(b - Aa)$ and the coefficient of friction.

5. AN EXTENDED PRINCIPLE OF GAUSS

An extended principle of Gauss is

$$(\ddot{q} - a - M^{-1}c)^{\mathrm{T}}M(\ddot{q} - a - M^{-1}c) = \min \tag{34}$$

subject to the restriction

$$A\ddot{q} = b. \tag{35}$$

Let us show that this also leads to Eq. (33). Introduce

$$y = M^{1/2}\left[\ddot{q} - a - M^{-1}c\right], \tag{36}$$

so that the extended Gauss principle becomes

$$y^{\mathrm{T}}y = \min \tag{37}$$

subject to

$$A\left[M^{-1/2}y + a + M^{-1}c\right] = b \tag{38}$$

or

$$AM^{-1/2}y = b - Aa - AM^{-1}c. \tag{39}$$

But the shortest length solution of the consistent linear algebraic equation system in Eq. (39) is

$$y = \left(AM^{-1/2}\right)^{+}\left(b - Aa - AM^{-1}c\right). \tag{40}$$

It follows from Eq. (36) that

$$\ddot{q} = M^{-1/2}y + a + M^{-1}c. \tag{41}$$

Thus

$$\ddot{q} = M^{-1/2}\left\{(AM^{-1/2})^{+}(b - Aa - AM^{-1}c)\right\} + a + M^{-1}c. \tag{42}$$

But this can be written as

$$\ddot{q} = a + M^{-1/2}(AM^{-1/2})^{+}(b - Aa) - M^{-1/2}(AM^{-1/2})^{+}AM^{-1}c + M^{-1}c, \tag{43}$$

$$\ddot{q} = a + M^{-1/2}(AM^{-1/2})^{+}(b - Aa) + M^{-1/2}\left\{I - (AM^{-1/2})^{+}(AM^{-1/2})\right\}M^{-1/2}c, \tag{44}$$

which is again the equation of motion of the system subject to constraint forces that do no work on the system and to constraint forces that do work on the system in a virtual displacement.

6. DISCUSSION

In this paper, we have presented an extension of Gauss's principle of least constraint and have exhibited the equation of motion to which it leads, Eq. (44). In modeling a mechanical system of mass points and rigid bodies the modeler may put $c = 0$, the classical choice, or choose $c \neq 0$ and specified by the work it does in a virtual displacement.

Reference

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